Recursively Updated Least Squares Based Modification Term for Adaptive Control

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Abstract— **We present an approach for combining standard recursive least squares based regression with proven direct model reference adaptive control using a recursively updated modification term. This approach is applicable to adaptive control problems where the uncertainty can be linearly parameterized. The combined training law drives the adaptive weights smoothly to a recursively updated least squares estimate of the ideal weights and is shown to have a stability proof. Expected improvement in performance of the adaptive law is validated through simulation.**

I. INTRODUCTION

This paper presents a combined direct and indirect approach to adaptive control for a class of adaptive control problems where the uncertainty can be linearly parameterized using a weighted combination of known nonlinear functions. The presented approach augments any baseline direct adaptive law by smoothly incorporating a recursively updated least squares estimate of the ideal weights through a modification term. The inclusion of the least squares based estimate provides additional information to the adaptive law and aids in convergence of the weights to optimal estimates of the ideal weights. The unique idea behind this approach is that it combines proven recursive least squares based methods such as Kalman filter based least squares regression [5] with standard gradient based adaptive laws while ensuring that the system states remain uniformly ultimately bounded.

In plants where the plant uncertainty can be linearly parameterized using a weighted combination of known nonlinear functions, the standard approach to the design of adaptive controllers is to use a weight training law that attempts to estimate the ideal weights in order to cancel the uncertainty. In the direct adaptive control methodology, adaptive laws are designed using a Lyapunov based approach and can be thought of as attempting to minimize a quadratic cost function of the instantaneous tracking error by using a gradient descent type method [9]. These adaptive laws however, are susceptible to local adaptation and weight drift [11],[4],[15]. Furthermore, it can be shown that adaptive laws formulated using the gradient descent methodology are always at most rank 1 [2]. In order to ensure that adaptive weights remain bounded, Ioannou et al. [4] suggested the use of a σ modification term which adds damping to the adaptive law. Narendra et al. [11] suggested the e modification term which uses the norm of the tracking error to scale the added damping. The methods are primarily designed to ensure boundedness of adaptive weights and they do little to ensure the convergence of the weights to ideal values. In order to improve weight convergence, Volyanskyy et al. suggested a novel Q modification [16] term which uses an integral over a moving window of the system uncertainty to improve weight convergence. Chowdhary and Johnson suggested the *Concurrent Learning* [2] method for incorporating long term learning in adaptive laws by using past and current data concurrently for adaptation. Concurrent learning constrains the weight update based on simultaneous gradient descent over a set of stored data point to the null space of the weight update based on current data. This augments the rank of the adaptive law while ensuring that it stays responsive to dynamic changes in the environment.

In this paper we maintain the idea of using past and current data concurrently for learning, however, we use optimal recursive least squares for adaptation based on past data combined with gradient descent for adaptation based on current data. This choice is motivated from the well known fact in linear algebra that the best linear fit for a given set of data can be obtained by solving the least squares problem [1]. The linear least squares regression problem can be viewed as a special case of the discrete Kalman filter and can be solved in a recursive way such that the update of the estimates become linear function of the error in the output [5]. Nguyen has shown improved tracking performance in direct adaptive controllers by using recursive least squares directly for weight adaptation [12]. We differ from Nguyen's approach by using recursive least squares to update a modification term rather than using it directly as the adaptive law. This allows us to augment any baseline law without affecting its responsiveness and other desirable properties. We show that inclusion of the modification term does not affect the uniform ultimate boundedness properties of the adaptive law. Furthermore, we show that the convergence of the weights to their ideal values is guaranteed if the system states are uniformly completely observable. The presented approach is recursive and hence conducive for online implementation. To that effect, we show the feasibility of using this approach on a wing rock dynamics model of a fixed wing aircraft. The simulation results confirm expected improvement in performance.

II. MODEL REFERENCE ADAPTIVE CONTROL

This section discusses the formulation of Model Reference Adaptive Control, the reader is referred to [9],[6],[8],[7] for further details. Let $x(t) \in \Re^n$ be the known state vector, let

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$u \in \Re$ denote the control input, and consider the following system where the uncertainty can be linearly parameterized:

$$\dot{x} = Ax(t) + B(u(t) + \Delta(x(t))), \quad (1)$$

where $A \in \Re^{n \times n}$, $B \in \Re^{n}$, and $\Delta(x)$ is a continuously differentiable function representing the scalar uncertainty. We assume that the pair A, B is completely controllable.

A reference model can be designed that characterizes the desired response of the system:

$$\dot{x}_{rm} = A_{rm}x_{rm}(t) + B_{rm}r(t), \quad (2)$$

where $A_{rm} \in \Re^{n \times n}$ is a Hurwitz matrix and $r(t)$ denotes a bounded reference signal. A tracking control law consisting of a linear feedback part $u_{pd} = Kx$, a linear feedforward part $u_{crm} = K_r[x_{rm}^T, r(t)]^T$, and an adaptive part $u_{ad}(x)$ is proposed to have the following form:

$$u = u_{crm} + u_{pd} - u_{ad}. \quad (3)$$

Define the tracking error e as $e(t) = x_{rm}(t) - x(t)$, with an appropriate choice of u_{crm} such that $Bu_{crm} = (A_{rm} - A)x_{rm} + B_{rm}r(t)$, the tracking error dynamics are found to have the form:

$$\dot{e} = A_m e + B(u_{ad}(x) - \Delta(x)), \quad (4)$$

where the baseline full state feedback controller $u_{pd} = Kx$ is assumed to be designed such that $A_m = A - BK$ is a Hurwitz matrix. Hence for any positive definite matrix $Q \in \Re^{n \times n}$, a positive definite solution $P \in \Re^{n \times n}$ exists to the Lyapunov equation:

$$A_m^T P + PA_m + Q = 0. \quad (5)$$

We now state the following assumptions:

Assumption 1: For simplicity, assume that $B = [0, 0, ..., 1]^T$ and $\Delta(x) \in \Re$.

Assumption 2: The uncertainty $\Delta(x)$ can be linearly parameterized, that is, there exist a vector of constants $W = [W_1, W_2, ..., W_m]^T$ and a vector of continuously differentiable functions $\Phi(x) = [\phi_1(x), \phi_2(x), ..., \phi_m(x)]^T$ such that

$$\Delta(x) = W^T \Phi(x). \quad (6)$$

This results in the following two cases:

Case 1: Structured Uncertainty: In this case the form of the linearly parameterized uncertainty is known, that is the mapping $\Phi(x)$ is known. In this case letting $\hat{W}$ denote the estimate W the adaptive law can be written as

$$u_{ad}(x) = \hat{W}^T \Phi(x). \quad (7)$$

On examining equation 4 we see that if $\hat{W} = W$ then the tracking error dynamics are globally exponentially stable. Hence, we desire to design an adaptive law such that $\hat{W}(t) \to W$ asymptotically.

For this case it is well known that the following adaptive law

$$\dot{\hat{W}} = -\Phi(x)e^T PB\Gamma_W \quad (8)$$

where Γ_W is a positive definite matrix of appropriate dimensions results in ultimate boundedness of the weights and $e(t) \to 0$ if the reference input $r(t)$ is persistently exciting [11],[4],[15]. Equation 8 will be referred to as the baseline adaptive law.

Case 2: Unstructured Uncertainty: In this more general case it is only known that the uncertainty $\Delta(x)$ can be linearly parameterized; hence the adaptive part of the control law is represented using a Radial Basis Function (RBF) Neural Network (NN):

$$u_{ad}(x) = \hat{W}^T \sigma(x). \quad (9)$$

where $\hat{W} \in \Re^l$ and $\sigma = [1, \sigma_2(x), \sigma_3(x), ..., \sigma_l(x)]^T$ is a vector of known radial basis functions. For $i = 2, 3..., l$ let c_i denote the RBF centroid and μ_i denote the RBF width then for each i The radial basis functions are given as:

$$\sigma_i(x) = e^{\|x - c_i\|^2/\mu_i}. \quad (10)$$

Appealing to the universal approximation property of RBF NN we have that given a fixed number of radial basis functions l there exists a vector of ideal weights $W^* \in \Re^l$ and a positive real number $\tilde{\epsilon}$ such that

$$\Delta(x) = W^{*T}\sigma(x) + \tilde{\epsilon}, \quad (11)$$

where $\tilde{\epsilon}$ can be made arbitrarily small given sufficient number of radial basis functions [13]. For a fixed number of radial basis functions, the control design task is to design an adaptive law such that $\hat{W}(t) \to W^*$ asymptotically. This will result in a uniform ultimate bound on the tracking error of equation 4 which is a function of $\tilde{\epsilon}$.

III. Recursive Least Squares based Combined Direct Indirect Adaptive Control

In this section we propose a combined direct and indirect adaptive law that uses a recursive optimal least squares estimate of the ideal weights for the linearly parameterized uncertainty. In the following, we will consider the case where the structure of the linearly parameterized uncertainty is exactly known (Case 1 in section II), the results will then be generalized to the case where it is only known that the uncertainty can be linearly parameterized (Case 2 in section II).

A. Least Squares Regression

We begin by describing a method by which least squares Regression can be performed online. Let N denote the number of available state measurements at time t, and θ denote an estimate of the ideal weighs W. For a given set of k samples, the model error $\Delta(k)$ can be observed by appealing to *assumption 1* and noting that:

$$\Delta(k) = B^T[\dot{x}(k) - Ax(k) - Bu(k)]. \quad (12)$$

Since $A, B, x(k), u(k)$ are known, the problem of estimating system uncertainty can be reduced to that of estimation of $\dot{x}$ by using 12. In cases where an explicit measurement for $\dot{x}$ is not available, for a given sample k, $\dot{x}$ can be estimated using an implementation of a fixed point smoother [3], the details of which are given in Appendix A.

Define the error $\epsilon(k) = \Delta(x(k)) - \Phi(x(k))^T\theta$, then the error for N discrete samples can be written in vector form as $\epsilon = [\epsilon(1), \epsilon(2), ..., \epsilon(N)]^T$. In order to arrive at the ideal estimate θ of the true weights W we must solve the following least squares problem:

$$\min_{W} \epsilon^T \epsilon. \tag{13}$$

Let $Y(k) = [\Delta(1), \Delta(2), ..., \Delta(k)]^T$ and define the following matrix:

$$X(k) = \begin{bmatrix} \phi_1(x(1)) & \phi_2(x(1)) & ... & \phi_m(x(1)) \\ \phi_1(x(2)) & \phi_2(x(2)) & ... & \phi_m(x(2)) \\ \\ \phi_1(x(k)) & \phi_2(x(k)) & ... & \phi_m(x(k)) \end{bmatrix}. \tag{14}$$

The solution to the above least squares problem can be obtained in a recursive form [5]. Alternatively, the solution to the least squares problem can be found through Kalman filtering theory by casting the least squares problem as parameter estimation problem. Appealing to assumption 2, we have the following model for the ideal weights θ,

$$\theta(k) = \theta(k-1), \tag{15}$$

$$\Delta(k) = \Phi^T(x(k))\theta(k). \tag{16}$$

Let $S(k)$ denote the Kalman filter error covariance matrix, $\hat{\theta}$ denote the estimate of the ideal weights θ, then setting the Kalman filter process noise covariance matrix $Q(k) = 0$, and the measurement covariance $R > 0$, the Kalman filter based least squares estimate can be updated in the following manner:

$$\hat{\theta}(k+1) = \hat{\theta}(k) + K(k+1)[\Delta(k+1) - \Phi^T(k+1)\hat{\theta}(k)], \tag{17}$$

$$K(k+1) = S(k)\Phi^T(k+1)[R + \Phi^T(k+1)S(k)\Phi(k+1)]^{-1}, \tag{18}$$

$$S(k+1) = [I - K(k+1)\Phi(k+1)]S(k). \tag{19}$$

B. Recursive Least Squares Based Modification

We now describe a method by which the least squares estimate of the ideal weights can be incorporated in the adaptive control law. Let $r^T = e^T PB$ where e, P, B are as in section II, Γ_W, Γ_θ are positive definite matrices denoting the learning rate. Let $\delta(t)$ denote the interval between two successive samples k and $k+1$, let T denote the time when sample k was obtained, for the current instant in time t, define the piece wise continuous sequence $\theta(t) = \hat{\theta}(k)$ for $T \le t < T + \delta(t)$, where $\hat{\theta}(k)$ is as in 17.

The adaptive law for weight estimates $\hat{W}$ is chosen as:

$$\dot{\hat{W}}(t) = -(\Phi(x(t))r^T(t) - \Gamma_\theta(\hat{W}(t) - \theta(t)))\Gamma_W. \tag{20}$$

In the above equation, the term $\Gamma_\theta(\hat{W}(t) - \theta(t)))$ serves to combine the indirect recursive least based estimate of the ideal weights smoothly into the baseline direct adaptive training law of equation 8. This term acts as a modification term to the baseline adaptive law.

In the following, we present Lyapunov based stability analysis for the chosen adaptive law.

Theorem 1 Consider the control law of equation 3, and the weight update law of 20, then all signals in the system 1 are bounded.

Proof: Let $\tilde{W} = \hat{W} - W$, tr denote the trace operator, and consider the following Lyapunov candidate function:

$$V(e, \tilde{W}) = \frac{1}{2}e^T Pe + \frac{1}{2}tr(\tilde{W}^T \Gamma_W{}^{-1}\tilde{W}). \tag{21}$$

The time derivative of the Lyapunov candidate along the system trajectory 4 is:

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Qe + r^T(\hat{W}^T\Phi(x) - W^T\Phi(x)) + tr(\dot{\hat{W}}\Gamma_W{}^{-1}\tilde{W}^T). \tag{22}$$

Let ϵ be such that $W = \theta + \epsilon$, adding and subtracting $(\hat{W}^T - \theta)^T\Gamma_\theta(\hat{W}^T - \theta)$ to equation 22 and using the definition of ϵ yields,

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Qe + r^T(\tilde{W}^T\Phi(x)) + tr(\dot{\hat{W}}\Gamma_W{}^{-1}\tilde{W}^T) + \tilde{W}^T\Gamma_\theta(\hat{W} - \theta) - \tilde{W}^T\Gamma_\theta(\hat{W} - \theta). \tag{23}$$

Rearranging yields:

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}e^T Qe + tr((\dot{\hat{W}}\Gamma_W{}^{-1} + \Phi(x)r^T + \Gamma_\theta(\hat{W} - \theta))\tilde{W}^T) - \tilde{W}^T\Gamma_\theta(\hat{W} - \theta). \tag{24}$$

Setting $tr((\dot{\hat{W}}\Gamma_W{}^{-1} + \Phi(x)r^T + \Gamma_\theta(\hat{W} - \theta))\tilde{W}^T) = 0$ yields the adaptive law. Consider the last term in 24,

$$\begin{aligned} \tilde{W}^T\Gamma_\theta(\hat{W} - \theta) = &\ (\hat{W} - W)^T\Gamma_\theta(\hat{W} - \theta) \\ = &\ (\hat{W} - W)^T\Gamma_\theta(\hat{W} - W) \\ &+ (\hat{W} - W)^T\Gamma_\theta\epsilon. \end{aligned} \tag{25}$$

Letting $\lambda_{min}(Q)$ and $\lambda_{min}(\Gamma_\theta)$ denote the minimum eigenvalues of Q and Γ_θ we have that equation 24 becomes:

$$\dot{V}(e, \tilde{W}) = -\frac{1}{2}\|e\|^2\lambda_{min}(Q) - \|\tilde{W}\|^2\lambda_{min}(\Gamma_\theta) - \tilde{W}^T\Gamma_\theta\epsilon. \tag{26}$$

With appropriate choice of $S(0)$ and R, the Kalman filter estimation error $\theta(k) - \hat{\theta}(k)$ and $S(k)$ of equation 17, 19 remain bounded, hence ϵ remains bounded. Therefore, for a given choice of Q and Γ_θ, $\dot{V}(e, \tilde{W}) < 0$ outside of a compact set, which shows that e and $\tilde{W}$ remain uniformly ultimately bounded, since A_{rm} is Hurwitz, x_{rm} is bounded, therefore it follows that x is bounded. This completes the proof.

Remark I : The above proof shows uniform ultimate boundedness of the tracking error and adaptive weights. It can be clearly seen that if $\epsilon \to 0$ then tracking error $e \to 0$. This condition will be achieved when $\theta \to W$, that is when the Kalman filter estimate of the ideal weights in 17 converges. The convergence of the Kalman filter estimate is related to choice of $S(0), R$ and uniform complete observability of the process model [3]. The uniform complete observability for the process model in 15 can be guaranteed if for some $T > 0$ and $\alpha > 0$ $\int_t^{t+T} \Phi(x(\tau))\Phi^T(x(\tau))d\tau \geq I\alpha$

Remark II : The above proof can be easily extended to the case where the structure of the uncertainty is unknown (case 2 from section II) by using Radial Basis Function Neural Networks for approximating the uncertainty. The following adaptive law will result in uniform ultimate boundedness of all states:

$$\dot{\hat{W}} = -(\sigma(x)r^T - \Gamma_\theta(\hat{W} - \theta))\Gamma_W. \tag{27}$$

Furthermore, referring to equation 11 and noting that in this case $\epsilon = \tilde{\epsilon}$, it can be shown that if the Kalman filter estimates of the ideal weights converge, then the weights will approach a neighborhood of the best linear approximation of the uncertainty.

Remark III : Although in this paper we restricted our attention to nonlinear systems which have a linear part and a scalar linearly parameterized uncertainty, it is straight forward to extend the analysis to more general nonlinear systems with the uncertainty in vector form as long as it is linearly parameterized.

Remark IV : The increased computational burden when using the adaptive law of equation 20 consists mainly of evaluating equations 17,18, and 19. It should be noted that since $\Phi(x) \in \Re^m$, the inversion in equation 18 is reduced to a division by a scalar.

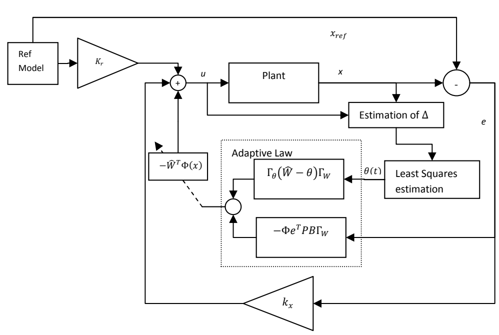

Fig. 1. Schematics of adaptive controller with recursive least squares Modification

Figure 1 shows the schematic of the presented adaptive control method with recursive least squares modification.

IV. Simulation results

In this section we use the method of Theorem 1 for the control a wing rock dynamics model.

Let ϕ denote the roll angle of an aircraft, p denote the roll rate, δ_a denote the aileron control input, then a model for wing rock dynamics is [10]:

$$\dot{\phi} = p \tag{28}$$
$$\dot{p} = \delta_a + \Delta(x). \tag{29}$$

Where $\Delta(x) = W_0 + W_1\phi + W_2 p + W_3|\phi|p + W_4|p|p + W_5\phi^3$. The parameters for wing rock motion are adopted from [14] and [16], they are $W_0 = 0.0, W_1 = 0.2314, W_2 = 0.6918, W_3 = -0.6245, W_4 = 0.0095, W_5 = 0.0214$. Initial conditions for the simulation are arbitrarily chosen to be $\phi = 1deg, p = 1deg/s$. The task of the controller is to drive the state to the origin. To that effect, a stable second order reference model is used with a natural frequency and a damping ratio of 1. The proportional gain K_x and the feedforward gain K_r in equation 3 are held constant for all of the presented simulation results.

The structure of the uncertainty and the ideal weights W are known for the wing rock dynamics model, hence the performance of the adaptive law can be accurately evaluated in terms of convergence of adaptive weights $\hat{W}$ to the ideal weights. The least squares problem is solved recursively using equations 17, 18, and 19. It is assumed that no *a priori* information is available about the ideal weights,hence we choose $\hat{\theta}(0) = 0$, consequently, the initial Kalman filter error covariance matrix $S(0)$ is chosen to have diagonal elements with large positive values. Figure 2 shows the performance of the baseline adaptive control law of equation 8 without the recursive least squares modification. The learning rate used was $\Gamma_W = 3$ for the low gain case, and $\Gamma_W = 10$ for the high gain case. It is seen that the performance of the controller in both cases is unsatisfactory. Figure 3 shows the phase portrait of the states when the combined direct and indirect adaptive law of Theorem 1 is used. It is seen that in both the low gain and the high gain case the system follows a smooth trajectory to the origin. Figure 4 shows the evolution of the adaptive control weights when only the baseline adaptive law of equation 8 is used. It is seen that the weights do not converge to the ideal values (W) and evolve in an oscillatory manner. In contrast, figure 5 shows the convergence of the weights when the combined direct and indirect adaptive law of Theorem 1 used. Figure 6 compares the reference model states with the plant states for the baseline adaptive law, while 7 compares the reference model and state output when the combined direct and indirect adaptive law is used. It can be seen that the performance of the adaptive law of Theorem 1 is superior to that of the baseline adaptive law. Although not shown in this paper, the adaptive law of Theorem 1 also showed improvement in control effort required by reducing the oscillation in the control input, and only a modest increase in computational efforts. Furthermore, we note that parameter convergence was observed despite using a non-persistently exciting reference input ($r(t) = 0 \forall t$), this is because $\Phi(x(t))$ was uniformly completely observable.

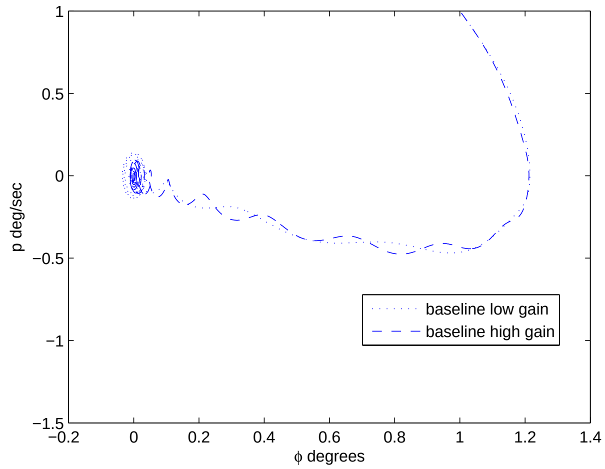

Fig. 2. Phase portrait of system states with only baseline adaptive control

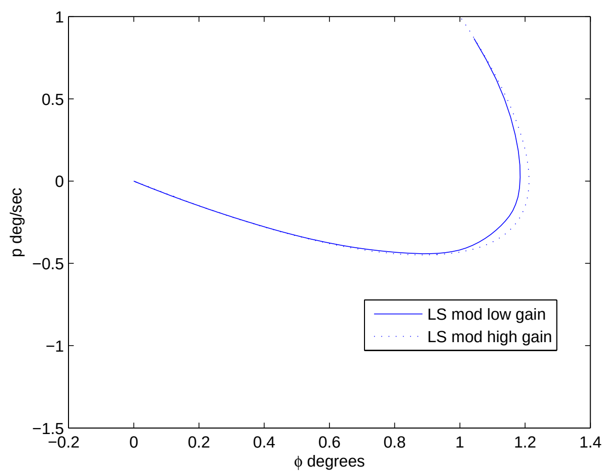

Fig. 3. Phase portrait of system states with combined direct and indirect adaptive controller

V. CONCLUSION

In this work we presented an approach to combined direct and indirect adaptive control for plants with a linearly parameterized uncertainty by augmenting a baseline direct adaptive law with recursive least squares. The presented method ensures that the adaptive weights arrive smoothly at the estimates of the ideal weights updated recursively using a Kalman filter based parameter estimator. We showed that the proposed combined adaptive law does not affect the uniform ultimate boundedness properties of the model reference adaptive controller. Furthermore, we indicated that if the Kalman filter estimates of the ideal weights converge, asymptotic stability can be guaranteed. We showed that the

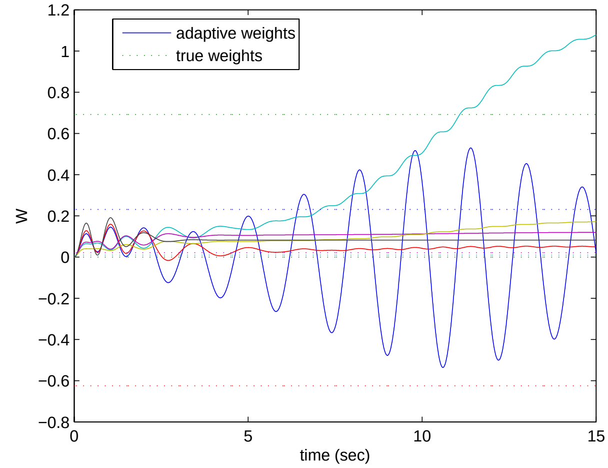

Fig. 4. Evolution of adaptive weights with only baseline adaptive control

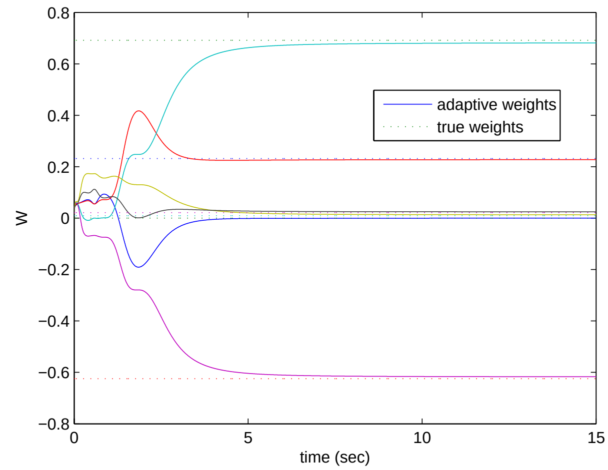

Fig. 5. Evolution of adaptive weights with combined direct and indirect adaptive controller

problem of estimating the system uncertainty required by the Kalman filter for estimating the ideal weights can be effectively reduced to that of estimating the state derivative $\dot{x}$ and suggested a time domain method for estimating the state derivative using optimal smoothing. Online optimal smoothing uses past as well as current data for generating a delayed estimate of $\dot{x}$. However, this delay does not affect the stability of the tracking error in the presented framework, since the indirect part of the adaptive law acts as a modification to a stable baseline direct adaptive law. An adaptive controller with the presented combined direct and indirect adaptive control modification was found to perform significantly better than the baseline adaptive controller for the exemplary problem of wing rock dynamics.

VI. ACKNOWLEDGMENTS

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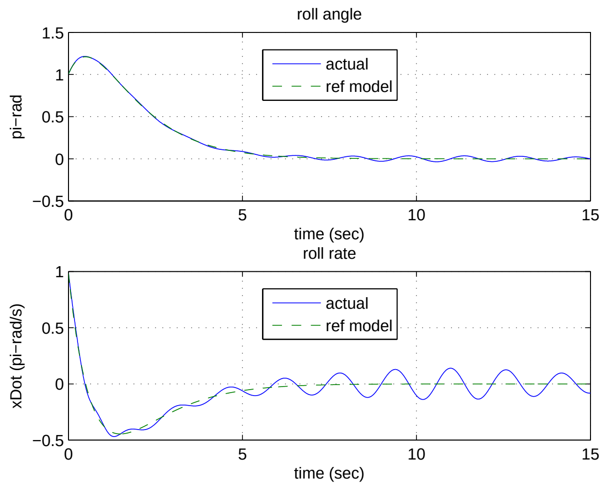

Fig. 6. Performance of adaptive controller with only baseline adaptive law

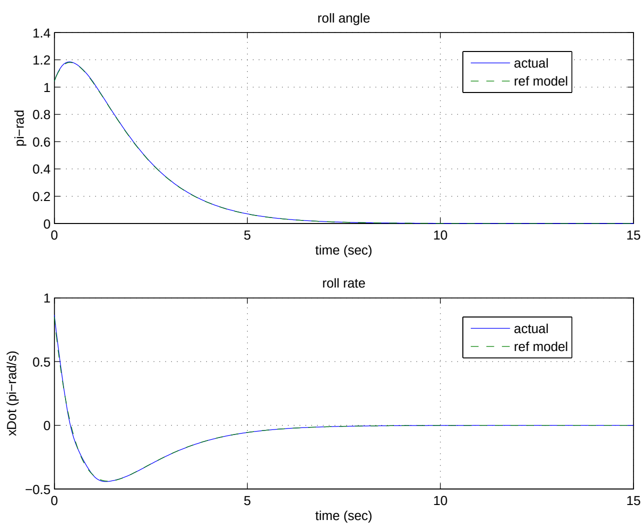

Fig. 7. Performance of combined direct and indirect adaptive controller

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VII. APPENDIX A: OPTIMAL FIXED POINT SMOOTHING

Numerical differentiation for estimation of state derivatives suffers from high sensitivity to noise. An alternate method is to use a Kalman filter based approach. Let x, be the state of the system and $\dot{x}$ be its first derivative, and consider the following system:

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{30}$$

Suppose x is available as sensor measurement, then optimal fixed point smoothing can be used for estimating $\dot{x}$ from available noisy measurements using the above system [3]. Optimal Fixed Point Smoothing is a non real time method for arriving at a state estimate at some time t, where $0 \leq t \leq T$, by using all available data up to time T. For ease of implementation on modern avionics, we present the relevant equations in the discrete form. Let $\hat{x}_{(k|N)}$ denote the estimate of the state $x = [\ x \quad \dot{x}\ ]^T$, let Z_k denote the measurements, $(-)$ denote predicted values, and $(+)$ denote corrected values, dt denote the discrete time step, Q and R denote the process and measurement noise covariance matrices respectively, while P denotes the error covariance matrix. Then the forward Kalman filter equations can be given as follow:

$$\Phi_k = e^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} dt}, \tag{31}$$

$$Z_k = [\ 1 \quad 0\ ] \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \tag{32}$$

$$\hat{x}_k(-) = \Phi_k \hat{x}_{k-1}, \tag{33}$$

$$P_k(-) = \Phi_k P_{k-1} {\Phi_k}^T + Q_k, \tag{34}$$

$$K_k = P_k(-) {H_k}^T [H_k P_k(-) {H_k}^T + R_k]^{-1}, \tag{35}$$

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[Z_k - H_k \hat{x}_k(-)], \tag{36}$$

$$P_k(+) = [I - K_k H_k] P_k(-). \tag{37}$$

Letting $\hat{x}_{k|k} = \hat{x}_k$ the smoothed state estimate can be given as:

$$\hat{x}_{k|N} = \hat{x}_{k|N-1} + B_N[\hat{x}_N(+) - \hat{x}_N(-)]. \tag{38}$$