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Supplement to

Concurrent Learning Adaptive Control for Systems with Unknown Sign of Control Effectiveness

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1 Introduction

This document is a supplement to the Conference on Decision and Control paper *Concurrent Learning Adaptive Control for Systems with Unknown Sign of Control Effectiveness* by Benjamin Reish and Girish Chowdhary. Due to page limits on conference materials, the following section had to be removed from the conference paper. The equation numbers apply to the aforementioned paper.

2 Stability of CL-MRAC in presence of Unknown Sign of Control Effectiveness and an Uncertain Allocation Matrix

Lyapunov stability criteria is used to demonstrate the ultimate boundedness of the zero solution per the following theorem.

Theorem 1 *Consider the system of* (1)*, the control law of* (3)*, the weight update laws of* (14)*,* (15)*, and* (16)*, and Theorem 1, then the zero solution of* $[e(t), \widetilde{B}(t), \widetilde{K}(t), \widetilde{K_r}(t)]$ *of the closed loop system is uniformly ultimately bounded.*

Proof. Focusing on a specific interval of time, $[t_i, t_{i+1}]$, allows the (t) to not be included for ease of reading. The Lyapunov candidate is chosen to be:

$$\begin{aligned} V(\zeta) =& \frac{1}{2}e^T P e + \frac{1}{2}tr(\widetilde{K}^T \Gamma_x^{-1} \widetilde{K}) + \frac{1}{2}tr(\widetilde{K_r}^T \Gamma_r^{-1} \widetilde{K_r}) \\ & \frac{1}{2}tr(\widetilde{B}^T \Gamma_B^{-1} \widetilde{B}) \end{aligned} \tag{17}$$

where $\zeta = [\begin{array}{cccc} e^T & \texttt{vec}(\widetilde{B})^T & \texttt{vec}(\widetilde{K})^T & \texttt{vec}(\widetilde{K_r})^T \end{array}]^T$ and $P, Q \in \mathbb{R}^{n \times n}$ and positive definite matrices such that $A_{rm}^T P + P A_{rm} = -Q$. The candidate Lyapunov function, (17), can be bounded above and below by

$$\begin{aligned} & \min\left(\lambda_{min}(P), \lambda_{min}(\Gamma_x^{-1}), \lambda_{min}(\Gamma_r^{-1}), \lambda_{min}(\Gamma_B^{-1})\right) ||\zeta||^2 \\ & \leq 2V(\zeta) \leq \\ & \max\left(\lambda_{max}(P), \lambda_{max}(\Gamma_x^{-1}), \lambda_{max}(\Gamma_r^{-1}), \lambda_{max}(\Gamma_B^{-1})\right) ||\zeta||^2 \end{aligned} \tag{18}$$

Let $[t_1, t_2, \ldots, t_{p_{max}}] \geq t_i < t_{i+1}$ be the sequence of times where each data point was recorded. Taking the derivative of the Lyapunov candidate along the system trajectory of (10) for each interval $[t_i, t_{i+1}]$ and simplifying:

$$
\begin{aligned}
\dot{V} =& - e^T Q e - e^T P \widetilde{B} \widetilde{K}^T x - e^T P \widetilde{B} \widetilde{K_r}^T r - tr\left[\widetilde{B}^T u u^T \widetilde{B}^T\right] \\
& - tr\left[\widetilde{K}^T \sum_{i=1}^{p} x_i \widehat{\epsilon}_{K\,i}^T\right] - tr\left[\widetilde{K_r}^T \sum_{j=1}^{p} r_j \widehat{\epsilon}_{Kr\,j}^T\right] \\
& - tr\left[\widetilde{B}^T \sum_{k=1}^{p} u_k u_k^T \widetilde{B}^T\right]
\end{aligned} \tag{19}
$$

Define $\widehat{\epsilon}_{Kr} = \Delta\epsilon_{Kr} + \epsilon_{Kr}$ and $\widehat{\epsilon}_K = \Delta\epsilon_K + \Delta\epsilon_{Kr} + \epsilon_K$ where ϵ_{Kr} is from (6), ϵ_K is from (5), and $\Delta\epsilon_K$ and $\Delta\epsilon_{Kr}$ are the differences caused by using $\widehat{B}$ instead of B. Define $\Omega_x = \sum_{i=1}^{p} x_i x_i^T$ and $\Omega_r = \sum_{j=1}^{p} r_j r_j^T$ which creates two non-negative matrices, and Ω_x is positive definite. Then

$$
\begin{aligned}
\dot{V} =& - e^T Q e - tr\left[\widetilde{K}^T \Omega_x \widetilde{K}\right] - tr\left[\widetilde{K_r}^T \Omega_r \widetilde{K_r}\right] \\
& - tr\left[\widetilde{B}^T \sum_{k=1}^{p} u_k u_k^T \widetilde{B}^T\right] - e^T P \widetilde{B} \widetilde{K}^T x - e^T P \widetilde{B} \widetilde{K_r}^T r \\
& - tr\left[\widetilde{K}^T \sum_{i=1}^{p} x_i \Delta\epsilon_{K\,i}^T + \widetilde{K}^T \sum_{i=1}^{p} x_i \Delta\epsilon_{Kr\,i}^T\right] \\
& - tr\left[\widetilde{K_r}^T \sum_{j=1}^{p} r_j \Delta\epsilon_{Kr\,j}^T\right] - tr\left[\widetilde{B}^T u u^T \widetilde{B}^T\right]
\end{aligned} \tag{20}
$$

The first four terms in (20) are negative definite and the last term is non-positive. Conservative bounds can be found by making the following substitutions. Let $c_Q = \lambda_{min}(Q)$ and $c_P = \|P\|$ where $\lambda_{min}(Q)$ returns the minimum eigenvalue of matrix Q. Now, A_{rm} of the reference model is chosen to be Hurwitz and the reference signal, r, is a scaled version of x_{des} which is bounded. Therefore, there exist scalars $c_{rm}, c_r > 0$ such that $c_{rm} > \|x_{rm}\|$ and $c_r > \|r\|$. The concurrent learning error terms, $\hat{\epsilon}$, (in all variants) are functions of x_i and r_j which are time invariant and bounded. Therefore, there exists scalars $c_{xerr}, c_{rerr}, c_u > 0$ such that $\|\sum_{i=1}^{p} x_i \Delta\epsilon_{K\,i}^T\| + \|\sum_{i=1}^{p} x_i \Delta\epsilon_{Kr\,i}^T\| < c_{xerr}$, and $c_{rerr} > \|\sum_{j=1}^{p} r_j \Delta\epsilon_{Kr\,j}^T\|$, and $c_u = \|\sum_{k=1}^{p} u_k u_k^T\|$. Since Ω_x and Ω_r are non-negative there exists scalars $c_{\Omega_x}, c_{\Omega_r} > 0$ such that $c_{\Omega_x} = \lambda_{min}(\Omega_x)$ and $c_{\Omega_r} = \lambda_{min}(\Omega_r)$. Lastly, by Assumption 2, matrices K^* and K_r^* exist then there exist scalars $\kappa_x, \kappa_r > 0$ such that $\kappa_x > \|K^*\|$ and $\kappa_r > \|K_r^*\|$. Then (20) can be bounded above using norms and the triangle inequality by:

$$
\begin{aligned}
\dot{V} \leq & - c_Q\|e\|^2 - \|\widetilde{B}\|^2\|e\|^2\|\widetilde{K}\|^2 - \kappa_x^2\|\widetilde{B}\|^2\|e\|^2 - c_{\Omega_r}\|\widetilde{K_r}\|^2 \\
& - c_u\|\widetilde{B}\|^2 - c_r^2\|\widetilde{B}\|^2\|\widetilde{K_r}\|^2 - \kappa_r^2 c_r^2\|\widetilde{B}\|^2 - c_{\Omega_x}\|\widetilde{K}\|^2 \\
& - c_{rm}^2\|\widetilde{B}\|^2\|\widetilde{K}\|^2 + 2\kappa_x c_{rm}^2\|\widetilde{K}\|\|\widetilde{B}\|^2 + \kappa_x^2 c_{rm}^2\|\widetilde{B}\|^2 \\
& + 2\kappa_x\|\widetilde{K}\|\|\widetilde{B}\|^2\|e\|^2 + 2c_{rm}\|\widetilde{B}\|^2\|\widetilde{K}\|^2 + 2\kappa_r c_r^2\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 4\kappa_x c_{rm}\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2\kappa_x^2 c_{rm}\|\widetilde{B}\|^2\|e\| + c_P\|e\|^2\|\widetilde{B}\|\|\widetilde{K}\| \\
& + 2c_r\|e\|\|\widetilde{B}\|^2\|\widetilde{K}\|\|\widetilde{K_r}\| + 2\kappa_x c_r\|e\|\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 2\kappa_r c_r\|e\|\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2\kappa_x\kappa_r c_r\|e\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\|\widetilde{K}\|\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_x\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\kappa_r\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_x\kappa_r\|\widetilde{B}\|^2 + c_{xerr}\|\widetilde{K}\| \\
& + c_P c_{rm}\|e\|\|\widetilde{B}\|\|\widetilde{K}\| + c_P c_r\|e\|\|\widetilde{B}\|\|\widetilde{K_r}\| + c_{rerr}\|\widetilde{K_r}\|
\end{aligned} \tag{21}
$$

For sufficiently large c_Q, c_{Ω_r}, c_{Ω_x}, and c_u, $\dot{V}(\zeta) \leq 0$ outside a compact set. To show that the set is compact, notice that (21) is quadratic in $\|e\|$, $\|\widetilde{B}\|$, $\|\widetilde{K}\|$, and $\|\widetilde{K_r}\|$. Solving these quadratic inequalities for one parameter while assuming the others are non-zero will derive conservative estimates for the positively invariant set where the solutions are bounded. Starting with $\|e\|$, assuming that $\|\widetilde{B}\| > 0$, $\|\widetilde{K}\| > 0$, and $\|\widetilde{K_r}\| > 0$, the bounds that force $\dot{V}$ to be negative are:

$$
\begin{aligned}
a_e = & - c_Q - \|\widetilde{B}\|^2\left(\|\widetilde{K}\|^2 + \kappa_x^2 - 2\kappa_x\|\widetilde{K}\|\right) + c_P\|\widetilde{B}\|\|\widetilde{K}\| \\
b_e = & \; c_P c_{rm}\|e\|\|\widetilde{B}\|\|\widetilde{K}\| + c_P c_r\|e\|\|\widetilde{B}\|\|\widetilde{K_r}\| + 2\kappa_x^2 c_{rm}\|\widetilde{B}\|^2 \\
& + 2c_r\|\widetilde{B}\|^2\|\widetilde{K}\|\|\widetilde{K_r}\| + 2\kappa_x c_r\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 2\kappa_r c_r\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2\kappa_x\kappa_r c_r\|\widetilde{B}\|^2 \\
c_e = & - c_{\Omega_r}\|\widetilde{K_r}\|^2 - c_u\|\widetilde{B}\|^2 - c_r^2\|\widetilde{B}\|^2\|\widetilde{K_r}\|^2 - \kappa_r^2 c_r^2\|\widetilde{B}\|^2 \\
& - c_{\Omega_x}\|\widetilde{K}\|^2 - c_{rm}^2\|\widetilde{B}\|^2\|\widetilde{K}\|^2 + 2\kappa_x c_{rm}^2\|\widetilde{K}\|\|\widetilde{B}\|^2 \\
& + \kappa_x^2 c_{rm}^2\|\widetilde{B}\|^2 + 2c_{rm}\|\widetilde{B}\|^2\|\widetilde{K}\|^2 + 2\kappa_r c_r^2\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 4\kappa_x c_{rm}\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\|\widetilde{K}\|\|\widetilde{K_r}\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\kappa_x\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_r\|\widetilde{K}\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\kappa_x\kappa_r\|\widetilde{B}\|^2 + c_{xerr}\|\widetilde{K}\| + c_{rerr}\|\widetilde{K_r}\| \\
\|e\| \geq & \frac{-b_e - \sqrt{b_e^2 - 4a_e c_e}}{2a_e}.
\end{aligned} \tag{22}
$$

Consider $\|\widetilde{B}\|$ next assuming that $\|e\| \geq 0$, $\|\widetilde{K}\| \geq 0$, and $\|\widetilde{K_r}\| \geq 0$. Then $\dot{V} \leq 0$

if:

$$\begin{aligned}
a_b =& - \|e\|^2\|\widetilde{K}\|^2 - \kappa_x^2\|e\|^2 - c_u - c_r^2\|\widetilde{K_r}\|^2 - \kappa_r^2 c_r^2 \\
& - c_{rm}^2\|\widetilde{K}\|^2 + 2\kappa_x c_{rm}^2\|\widetilde{K}\| + \kappa_x^2 c_{rm}^2 + 2\kappa_x\|\widetilde{K}\|\|e\|^2 \\
& + 2c_{rm}\|\widetilde{K}\|^2 + 2\kappa_r c_r^2\|\widetilde{K_r}\| + 4\kappa_x c_{rm}\|\widetilde{K}\| \\
& + 2\kappa_x^2 c_{rm}\|e\| + 2c_r\|e\|\|\widetilde{K}\|\|\widetilde{K_r}\| + 2\kappa_x c_r\|e\|\|\widetilde{K_r}\| \\
& + 2\kappa_r c_r\|e\|\|\widetilde{K}\| + 2\kappa_x\kappa_r c_r\|e\| + 2c_r c_{rm}\|\widetilde{K}\|\|\widetilde{K_r}\| \\
& + 2c_r c_{rm}\kappa_x\|\widetilde{K_r}\| + 2c_r c_{rm}\kappa_r\|\widetilde{K}\| + 2c_r c_{rm}\kappa_x\kappa_r \\
b_b =& \, c_P\|e\|^2\|\widetilde{K}\| + c_P c_{rm}\|e\|\|\widetilde{K}\| + c_P c_r\|e\|\|\widetilde{K_r}\| \\
c_b =& - c_Q\|e\|^2 - c_{\Omega_r}\|\widetilde{K_r}\|^2 - c_{\Omega_x}\|\widetilde{K}\|^2 + c_{xerr}\|\widetilde{K}\| \\
& + c_{rerr}\|\widetilde{K_r}\|
\end{aligned}$$

$$\|\widetilde{B}\| \geq \frac{-b_b - \sqrt{b_b^2 - 4a_b c_b}}{2a_b} \tag{23}$$

Thirdly, consider $\|\widetilde{K}\|$ assuming that $\|e\| \geq 0$, $\|\widetilde{B}\| \geq 0$, and $\|\widetilde{K_r}\| \geq 0$, then $\dot{V} \leq 0$ if:

$$\begin{aligned}
a_x =& - c_{\Omega_x} - \|\widetilde{B}\|^2\|e\|^2 - c_{rm}^2\|\widetilde{B}\|^2 + 2c_{rm}\|\widetilde{B}\|^2 \\
b_x =& \, 2\kappa_x c_{rm}^2\|\widetilde{B}\|^2 + 2\kappa_x\|\widetilde{B}\|^2\|e\|^2 + 4\kappa_x c_{rm}\|\widetilde{B}\|^2 \\
& + c_P\|e\|^2\|\widetilde{B}\| + 2c_r\|e\|\|\widetilde{B}\|^2\|\widetilde{K_r}\| + 2\kappa_r c_r\|e\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_r\|\widetilde{B}\|^2 \\
& + c_P c_{rm}\|e\|\|\widetilde{B}\| + c_{xerr} \\
c_x =& - c_Q\|e\|^2 - \kappa_x^2\|\widetilde{B}\|^2\|e\|^2 - c_{\Omega_r}\|\widetilde{K_r}\|^2 - c_u\|\widetilde{B}\|^2 \\
& - c_r^2\|\widetilde{B}\|^2\|\widetilde{K_r}\|^2 - \kappa_r^2 c_r^2\|\widetilde{B}\|^2 + \kappa_x^2 c_{rm}^2\|\widetilde{B}\|^2 \\
& + 2\kappa_r c_r^2\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2\kappa_x^2 c_{rm}\|\widetilde{B}\|^2\|e\| \\
& + 2\kappa_x c_r\|e\|\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2\kappa_x\kappa_r c_r\|e\|\|\widetilde{B}\|^2 \\
& + 2c_r c_{rm}\kappa_x\|\widetilde{K_r}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_x\kappa_r\|\widetilde{B}\|^2 \\
& + c_P c_r\|e\|\|\widetilde{B}\|\|\widetilde{K_r}\| + c_{rerr}\|\widetilde{K_r}\| \\
\|\widetilde{K}\| \geq & \frac{-b_x - \sqrt{b_x^2 - 4a_x c_x}}{2a_x}
\end{aligned} \tag{24}$$

And, lastly, consider $\|\widetilde{K_r}\|$, assuming that $\|e\| \geq 0$, $\|\widetilde{B}\| \geq 0$, and $\|\widetilde{K}\| \geq 0$, then

$\dot{V} \leq 0$ if:

$$\begin{aligned}
a_r =& -c_{\Omega_r} - c_r^2\|\widetilde{B}\|^2\\
b_r =&\ 2\kappa_r c_r^2\|\widetilde{B}\|^2 + 2c_r\|e\|\|\widetilde{B}\|^2\|\widetilde{K}\| + 2\kappa_x c_r\|e\|\|\widetilde{B}\|^2\\
&+ 2c_r c_{rm}\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_x\|\widetilde{B}\|^2\\
&+ c_P c_r\|e\|\|\widetilde{B}\| + c_{rerr}\\
c_r =& -c_Q\|e\|^2 - \|\widetilde{B}\|^2\|e\|^2\|\widetilde{K}\|^2 - \kappa_x^2\|\widetilde{B}\|^2\|e\|^2 - c_u\|\widetilde{B}\|^2\\
&- \kappa_r^2 c_r^2\|\widetilde{B}\|^2 - c_{\Omega_x}\|\widetilde{K}\|^2 - c_{rm}^2\|\widetilde{B}\|^2\|\widetilde{K}\|^2\\
&+2\kappa_x c_{rm}^2\|\widetilde{K}\|\|\widetilde{B}\|^2 + \kappa_x^2 c_{rm}^2\|\widetilde{B}\|^2 + 2c_{rm}\|\widetilde{B}\|^2\|\widetilde{K}\|^2\\
&+ 2\kappa_x\|\widetilde{K}\|\|\widetilde{B}\|^2\|e\|^2 + 4\kappa_x c_{rm}\|\widetilde{K}\|\|\widetilde{B}\|^2\\
&+ 2\kappa_x^2 c_{rm}\|\widetilde{B}\|^2\|e\| + c_P\|e\|^2\|\widetilde{B}\|\|\widetilde{K}\| + c_{xerr}\|\widetilde{K}\|\\
&+ 2\kappa_r c_r\|e\|\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2\kappa_x\kappa_r c_r\|e\|\|\widetilde{B}\|^2\\
&+ 2c_r c_{rm}\kappa_r\|\widetilde{K}\|\|\widetilde{B}\|^2 + 2c_r c_{rm}\kappa_x\kappa_r\|\widetilde{B}\|^2\\
&+ c_P c_{rm}\|e\|\|\widetilde{B}\|\|\widetilde{K}\|\\
\|\widetilde{K_r}\| \geq& \frac{-b_r - \sqrt{b_r^2 - 4a_r c_r}}{2a_r}
\end{aligned} \tag{25}$$

The compact set outside of which $\dot{V}(\zeta) \leq 0$ is described by (22), (23), (24), and (25). Since (17) is a common Lyapunov candidate [1] across all time intervals $[t_i, t_{i+1}]$, the system $[e, \widetilde{B}, \widetilde{K}, \widetilde{K_r}]$ is uniformly ultimately bounded because all solutions will eventually arrive in the set previously described. ■

Bibliography

[1] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.